数理解析研究所講究録
181巻 1973年 86-91



# Mathematical Properties of Responses of a Neuron Model

## A System as a Rational Number Generator

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**Abstract**

Recently, Nagumo and Sato proposed a mathematical neuron model in the form of a nonlinear difference equation and investigated its response characteristic. The result showed that the input-output relationship of the neuron model is quite complicated and takes the form of an extended Cantor's function. It also explained the "unusual and unsuspected" phenomenon found by Harmon in experimental studies with his transistor neuron model. – In this paper, a fraction representation of a sequence of pulses is proposed. A mathematical treatment of the same neuron model based on the representation gives the same result as in the previous paper. Moreover, many mathematical properties, including the one where the ratio of the number of 1's contained in a cycle of a sequence to the length of the cycle gives any rational number between 0 and 1, were obtained by investigating sequences generated by the model.

### I. Introduction

In the previous paper [2], we proposed a mathematical model of a neuron from a functional point of view. This model is represented by a nonlinear difference equation as follows:

$$u_{n+1} = 1\left[A_n - c\sum_{r=0}^{n} b^{-r} u_{n-r} - \theta\right], \tag{1}$$

$$1[\xi] = \begin{cases} 1 & \xi \geqq 0 \\ 0 & \xi < 0, \end{cases}$$

where $u_n$ represents the state of a neuron at the instant $n$, namely, $u_n = 0$ denotes the resting state and $u_n = 1$ the excited state, $A_n$ is the magnitude of the input stimulus applied at the instant $n$, $\theta$ the threshold value, and $c > 0$, $b > 1$. The term $c\sum_{r=0}^{n} b^{-r} u_{n-r}$ represents the influence of the past states of the neuron to the state at the next instant.

By investigating the mathematical model, we clarified the relationship between the magnitude of the input stimulus applied to the neuron and the output which gives a sequence of pulses, and also succeeded in explaining an "unusual and unsuspected" phenomenon which was found by Harmon in experimental studies with his artificial neurons [1].

In this paper we will analyze the same mathematical model by a different method, which gives eventually the same result as before. Furthermore, mathematical properties shown by the model, especially the one which seems to be interesting in number theory, will be reported.

In what follows, any pulse sequence corresponds uniquely to a rational number between 0 and 1. And hence, we can deal with the number instead of the sequence of the pulses. This method seems to be also useful in analyzing the output behaviour of networks consisting of many neurons.

### II. Fraction Representation of a Pulse Sequence

II–1. Let us consider a sequence of 0's and 1's. A sp. 1 means the existence of a pulse and a 0 its non-existence in the outputs of the neuron model. Thus a sequence of pulses is represented by terms of a sequence of numbers:

$$a_1, a_2, \ldots, a_n \ (a_i = 1 \text{ or } 0).$$

Let a number $x$ correspond to the pulse sequence as follows:

$$x = a_n \cdot \frac{1}{2} + a_{n-1} \cdot \frac{1}{2^2} + \cdots + a_{n-i} \cdot \frac{1}{2^{i+1}} + \cdots, \tag{2}$$

where $a_{n-i} = 0$ for $n - i \leqq 0$.

For instance, the sequence

$$\ldots 0101 \quad \ldots 01$$

corresponds to the number

$$1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2^2} + 1 \cdot \frac{1}{2^3} + \cdots = \frac{2}{3}$$

by Eq. (2), and vice versa, the number $\frac{2}{3}$ corresponds to a repetitive sequence of 01 with a present value 1.* Clearly $\frac{1}{3}$ corresponds to the same sequence with a present value 0.

Kybernetik 11, 208—216 (1972)

One may consider that a sequence with period 2 should correspond to a number independent of its present value. However, the next state of a neuron is affected by its past history as described by Eq. (1). The more time goes back to the past, the less becomes the influence of the past states on the next one. Therefore, a sequence has different meanings on the next state of a neuron according to its present state.

II–2. Fraction representation of a periodic sequence.

As described in the above, a sequence will correspond to several numbers depending on the present state.

Let us consider numbers corresponding to a sequence with period $m$:

$$\ldots a_1 a_2 \ldots a_m a_1 a_2 \ldots a_m .$$

We have $m$ different numbers according to the present states:

$$\alpha_m = \frac{a_m}{2} + \frac{a_{m-1}}{2^2} + \cdots + \frac{a_1}{2^m} + \frac{a_m}{2^{m+1}} + \cdots$$
$$= \frac{1}{2^m - 1} a_m a_{m-1} \ldots a_1(2),$$
$$\alpha_{m-1} = \frac{a_{m-1}}{2} + \frac{a_{m-2}}{2^2} + \cdots + \frac{a_1}{2^{m-1}} + \frac{a_m}{2^m} + \cdots$$
$$= \frac{1}{2^m - 1} a_{m-1} \ldots a_1 a_m(2),$$
$$\ldots \qquad \ldots$$
$$\alpha_1 = \frac{a_1}{2} + \frac{a_m}{2^2} + \cdots + \frac{a_2}{2^m} + \cdots$$
$$= \frac{1}{2^m - 1} a_1 a_m \ldots a_2(2),$$

where a number 2 in parentheses means the binary representation, which will be omitted.

The canonical cycle of the above sequence is defined to be:

$a_1 a_2 \ldots a_m$.

In what follows, by $a_1 a_2 \ldots a_m$ we sometimes represent not only the canonical cycle but a repetetive sequence of it.

### III. Modification of the Neuron Model

The term $c \sum_{r=0}^{n} b^{-r} u_{n-r}$ is a function of the present and past states of the neuron:

$$u_0, u_1, \ldots, u_n .$$

Writing $u_i = 0$ for $i \leqq 0$, we obtain an equality:

$$c \sum_{r=0}^{n} b^{-r} u_{n-r} = c \sum_{r=0}^{\infty} b^{-r} u_{n-r} .$$

And assigning a sequence $\ldots u_0 u_1 \ldots u_n$ to a number $x_n$ by Eq. (2), we can put

$$c \sum_{r=0}^{\infty} b^{-r} u_{n-r} \equiv h(x_n) , \tag{3}$$

where $h(x)$ is clearly a monotone-increasing function of $x$ because $b > 1$. From Eqs. (1) and (3), we have

$$u_{n+1} = 1\,[d_n - h(x_n)] , \tag{4}$$

where $d_n = A_n - \theta$.

Or, instead of Eq. (4) which gives output pulse sequences of the model, we have a set of difference equations:

$$\left.\begin{aligned} x_{n+1} &= \frac{1}{2} + \frac{1}{2} x_n, \ d_n - h(x_n) \geqq 0, \\ x_{n+1} &= \frac{1}{2} x_n \qquad , \ d_n - h(x_n) < 0, \end{aligned}\right\} \tag{5}$$

which gives a sequence of the corresponding numbers.

Let us consider the case that $d_n \equiv d$, i.e., the magnitude of the input stimulus is constant with respect to time. Since the function $h(x)$ is a monotone-increasing function of $x$, the equation

$$d - h(x) = 0$$

has only one solution, which is denoted by $\xi_0$.

III–1. The case that $\xi_0 \geqq 1$.

For all $x (0 \leqq x \leqq 1)$,

$$d - h(x) \geqq 0 .$$

Consequently, from Eq. (5),

$$x_\infty = \frac{1}{2} + \frac{1}{2} x_\infty$$

at $n \to \infty$, which gives

$$x_\infty = 1 = .111 \ldots\ldots (2) .$$

This means that the neuron fires at any instant.

III–2. The case that $\xi_0 \leqq 0$.

If $\xi_0 < 0$, $d - h(x) < 0$ for all $x (0 \leqq x \leqq 1)$. Thus we have a sequence $\ldots 00 \ldots 0$. This means that the neuron is always in the resting state.

If $\xi_0 = 0$, Eq. (5) gives a sequence $\{x_n\}$ which converges to zero as $n \to \infty$ for any initial value of $x_n$, which means that we have consequently a neuron in the resting state for $\xi_0 = 0$.

III–3. The case that $0 < \xi_0 < 1$.

A number of firing modes will be obtained in this case as reported in the previous paper. Before investigating the case again, we will give some preparatory discussions.

## IV. Concept of "Node of Period $m$" and Euler's Function

IV–1. Construction of nodes and periodic sequences.

Step 1. Let us take two different points on line. These points are called nodes of period 1 and denoted by $n_1^1$ and $n_2^1$. The value $m=1$ is given to both nodes. At the same time, repetitive sequences of 0 and 1 are assigned to $n_1^1$ and $n_2^1$, respectively. Namely a sequence $\ldots 00 \ldots 0$ corresponds to the node $n_1^1$ and $\ldots 1 \ldots 1$ to the node $n_2^1$.

Step 2. $m \leftarrow m+1$ ($m$ is increased by 1).

Step 3. The two values $i_1$ and $i_2$ given to the nodes $n_{k_1}^{i_1}$ and $n_{k_2}^{i_2}$, each of which is adjacent to the other on the line segment, are summed. If the result is equal to $m$, a node $n_k^m$ of period $m$ is newly made between the two nodes $n_{k_1}^{i_1}$ and $n_{k_2}^{i_2}$. There are at least two nodes of period $m$, for $n_1^m$ exists between $n_1^1$ and $n_1^{m-1}$, and $n_{\varphi(m)}^m$ between $n_{\varphi(m-1)}^{m-1}$ and $n_2^1$, where $m \geqq 3$ and $\varphi(m)$ means the number of nodes of period $m$.

Let a sequence with a canonical cycle

$$l_k^m = l_{k_1}^{i_1} \cdot l_{k_2}^{i_2} \qquad (\,\cdot\, : \text{concatenation})$$

correspond to the node of period $m$, where

$$k = 1, 2, \ldots, \varphi(m)$$

and $l_{k_1}^{i_1}$ and $l_{k_2}^{i_2}$ means the canonical cycles of sequences corresponding to nodes $n_{k_1}^{i_1}$ and $n_{k_2}^{i_2}$, respectively.

For instance, since

$$l_1^1 = 0,\ l_1^{m-1} = 0^{(m-2)}1,\ l_{\varphi(m-1)}^{m-1} = 0\,1^{(m-2)}$$

and

$$l_2^1 = 1,$$

we have successively

$$l_1^m = 0^{(m-1)}1 = \underbrace{0\,0\,0 \ldots 0}_{(m-1)}\ 1$$

and

$$l_{\varphi(m)}^m = 0\ 1^{(m-1)} = 0\underbrace{1\,1 \ldots 1}_{(m-1)}.$$

Step 4. Repeat steps 2 and 3.

The following few steps make the construction method clear.

Two nodes exist at step 1.

| | |
|---|---|
| 1 | 1 |
| (0) | (1) |

Nodes and the corresponding canonical cycles at step 2 are:

| | | |
|---|---|---|
| 1 | 2 | 1 |
| (0) | (01) | (1), |

where the cycles are shown in the parentheses.

At step 3:

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 2 | 3 | 1. |
| (0) | (001) | (01) | (011) | (1) |

At step 4:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 4 | 3 | 2 | 3 | 4 | 1 |
| (0) | (0001) | (001) | (01) | (011) | (0111) | (1) |

At step 5:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4 | 3 | 5 | 2 | 5 | 3 | 4 | 5 | 1 |
| | (00001) | | | (00101) | | (01011) | | | (01111) | |

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Two sequences are called adjacent to each other at some step if the corresponding nodes are adjacent to each other on the line at the same step.

IV–2. On the number of nodes at each step.

We have the following theorem.

*Theorem 1*

Let us denote the number of nodes of period $m$ by $\varphi(m)$. $\varphi(m)$ is nothing but Euler's function [3], namely, $\varphi(m)$ gives the number of positive integers prime to and less than $m$. ◀

The proof of Theorem 1 will be given in the Appendix (Prop. 10).

IV–3. Number of 1's included in a cycle.

*Theorem 2*

Let $u$ denote the number of 1's included in a cycle $a_1 a_2 \ldots a_m$ of a periodic sequence constructed by the method of IV–1. Then $m$ and $u$ are prime to each other. ◀

The proof will be given in the Appendix (Prop. 2).

## V. On the Existence of the Periodic Sequences

In what follows, we will give theorems which assure the existence of periodic sequences constructed by the method. By the term "existence" of a sequence, we mean that the system described by Eq. (1) can give it as an output for a certain magnitude of input stimulus.

For any sequence $a_1 a_2 \ldots a_m$ constructed by the method, we define an interval $I_m = [\alpha_{m-1}, \alpha^*]$, where

$$\alpha_{m-1} = \frac{1}{2^m - 1} a_{m-1} \ldots a_1 a_m$$

and

$$\alpha^* = \frac{2}{2^m - 1} a_1 a_2 \ldots a_m .$$

We will discuss in the Appendix (Prop. 16) that the interval is not meaningless; that is $\alpha_{m-1} < \alpha^*$.

*Theorem 3*

A necessary and sufficient condition under which a repetitive sequence of $a_1 a_2 \dots a_m$ exists in the sense mentioned above is that the solution $\xi_0$ of the equation

$$\alpha - h(x) = 0$$

belongs to the interval $I_m = [\alpha_{m-1}, \alpha^*]$. ◀

The proof will be given in the Appendix (Prop. 18).

*Theorem 4*

If two sequences whose canonical cycles are $a_1 a_2 \dots a_m$ and $b_1 b_2 \dots b_n$ exist adjacently, then a composed sequence with a canonical cycle

$$a_1 a_2 \dots a_m b_1 \dots b_n$$

can exist. ◀

Theorem 4 cites that all sequences constructed by the method mentioned in IV–1 can be realized as outputs of the neuron model represented by Eq. (1).

Next let us define a function $F(x)$ as follows:

For any $x$ in $I_m = [\alpha_{m-1}, \alpha^*]$,

$$F(x) = \frac{\text{number of 1's in } a_1 a_2 \dots a_m}{m \text{ (length of the cycle)}},$$

where $a_1 a_2 \dots a_m$ is the sequence corresponding to $I_m$.

Then we have the following theorems.

*Theorem 5*

$F(x)$ can take any rational number between 0 and 1. ◀

*Theorem 6*

$F(x)$ is a flat, continuous almost everywhere and nondecreasing function of $x (0 \leq x \leq 1)$. ◀

These theorems will be proved in the Appendix. Many other mathematical properties obtained from the output sequences of the neuron model will be given in the Appendix.

*Ex. 1*

Two sequences with period 1; $\dots 00 \dots 0$ and $\dots 11 \dots 1$ exist as shown in III–1 and III–2. Accordingly, the repetitive sequences of 01 can exist. The condition of the existence is that the solution $\xi_0$ of the equation $d - h(x) = 0$ belongs to the interval

$$I_2 = \left[\frac{1}{3}, \frac{2}{3}\right],$$

because

$$\alpha_{2-1} = \alpha_1 = \frac{1}{2^2 - 1} 01 = \frac{1}{3}$$

and

$$\alpha^* = \frac{2}{2^2 - 1} 01 = \frac{2}{3}.$$

*Ex. 2*

Two sequences of 001 and 011 can exist.

Since 001 is the concatenation of 0 and 01, we have

$$\gamma_{1+2-1} = \gamma_2 = \frac{1}{2^2 - 1} 001 = \frac{1}{7}$$

and

$$\gamma_{2-1} = \gamma_1 = \frac{1}{2^3 - 1} 010 = \frac{2}{7} \left(\gamma^* = \frac{2}{2^3 - 1} 001 = \frac{2}{7}\right).$$

Consequently, the repetitive sequence of 001 is realizable by the model if the solution lies in $[\frac{1}{7}, \frac{2}{7}]$. In the same way, the interval $[\frac{5}{7}, \frac{6}{7}]$ corresponds to the sequence of 011.

*Ex. 3*

The sequences 0110110111 · 01101101110110111 can exist at a node between the nodes of the sequences 0110110111 and 01101101110110111.

0110110111:

$$\alpha_{m-1} = \alpha_9 = \frac{1}{2^{10} - 1} 1101101101 = \frac{1555(8)}{1777(8)},$$

$$\alpha^* = \frac{2}{2^{10} - 1} 0110110111 = \frac{1556(8)}{1777(8)},$$

$$I_m = \left[\frac{1555(8)}{1777(8)}, \frac{1556(8)}{1777(8)}\right].$$

01101101110110111:

$$\beta_{n-1} = \beta_{16} = \frac{333555(8)}{377777(8)},$$

$$\beta^* = \frac{333556(8)}{377777(8)},$$

$$I_n = \left[\frac{333555(8)}{377777(8)}, \frac{333556(8)}{377777(8)}\right].$$

0110110111 · 01101101110110111:

$$\gamma_{m+n-1} = \gamma_{26} = \frac{667333555(8)}{777777777(8)},$$

$$\gamma^* = \frac{667333556(8)}{777777777(8)},$$

$$I_{m+n} = [\gamma_{26}, \gamma^*],$$

$$F(x) = \frac{7}{10} \quad \text{for any } x \text{ in } I_m,$$

$$F(x) = \frac{12}{17} \quad \text{for any } x \text{ in } I_n,$$

$$F(x) = \frac{19}{27} \quad \text{for any } x \text{ in } I_{m+n}.$$

## References


1. Harmon, L. D.: Studies with artificial neurons, I: Properties and functions of an artificial neuron. Kybernetik, 1, 89–101 (1961),
2. Nagumo, J., Sato, S.: On a response characteristic of a mathematical neuron model, Kybernetik. 10, 155–164 (1972).
3. Tichmarsh, E. C.: The theory of functions. Oxford: Clarrendon Press 1968.

西尾：最初のモデルでいいですが，各系列に対して区間が決りますね。それのある1つの区間に，$\xi_0$ がおちる回路のなかで，$\sum_{r=0} e^{-r}u_{n-r}$ の $r$ をうごかす範囲が一番少くてすむ，つまり，2進数表現をしたときに一番短い，あるいは，最短メモリーといってもいいと思いますが，そのような回路に関する研究はしていますか。

佐藤：していません。

西尾：この研究は非常に面白いと思うのですが…。オートノーマスな回路で複雑な系列をつくるとき，サイクリックな系列しかつくれません。有限オートマトンで最終的にはサイクリックになるというのですが，このモデルの回路では，最初からサイクリックというので私は非常にきれいな結果がでたと思います。ところで，そのサイクリック系列を出すのに，自分の内部にどれだけのメモリが必要か。そのメモリの最小な回路がほしい。つまり，この区間のなかの数を全部2進数展開し，一番短い桁で表現きる数がある。その数の桁はいくらか，ということです。その答がでればよいと思うのです。

あとの方のモデル，0AAというような入力を入れる場合は相当むずかしそうですから最初の方だけでも結構ですが…。

山口(昌)：θがゼロでない場合はむずかしいですか。

佐藤：やはりかなりむずかしいと思う。

南雲：5枚目のビラ，性質15ですが，F(x)は区間のxに対してだけ定義されている関数ですね。ところで性質17の関数F(x)は，15のF(x)とは違いますね。ここで議論される区間は，[0, 1]にどこにもdenseにあります。ですから，性質15のF(x)は，一意に[0, 1]の区間全体で定義された，あるF(x)に拡張できるわけです。性質17ではその関数のことを述べているわけですね。そして，その関数は（ビラでの書き方はおかしいと思いますが），連続であって，ほとんどいたるところでflatです。

それから，区間の長さが合計して1になるということから，他の系列は出現しないといったけれども，そうではなくて，他の周期系列がでないのであって，周期系列でないようなのは何が出現するか分らないということです。

佐藤：そうです。